A proof of the Gregory–Leibniz series and new series for calculating π

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Abstract

A non-traditional proof of the Gregory-Leibniz series, based on the relationships among the zeta function, Bernoulli coefficients, and the Laurent expansion of the cotangent is given. New series for calculating π are obtained.

1. Introduction

There are infinitely many primes in each of the four arithmetic progressions $4n \pm 1$ and $6n \pm 1$ [Dav]. Every twin prime pair $(p, p + 2)$, excluding $(3, 5)$, is of the form $(6n - 1, 6n + 1)$. The Gregory-Leibniz series

$$\pi/4 = 1 - (1/3) + (1/5) - (1/7) + (1/9) - (1/11) + (1/13) - \ldots \tag{1.1}$$

can be rearranged so that products of the form $(4n + 1)(4n - 1)$ occur as denominators

$$\pi/4 = 1 - 2\{(1/15) + (1/63) + (1/143 + (1/255) + \ldots\}$$

$$= 1 - 2 \sum_{n=1}^{\infty} 1/((4n)^2 - 1) \tag{1,2}$$

Alternatively, the products of every twin prime pair, excluding $(3, 5)$, appear as denominators in the form:

$$\pi/4 = 1 - (1/3) + (2/(5 \cdot 7)) + 1/9 - (2/(11 \cdot 13)) - 1/15 + (2/(17 \cdot 19)) + \ldots$$

$$= (\sqrt{3}/2)[1 - 2 \sum_{n=1}^{\infty} 1/((6n)^2 - 1)]. \quad \text{(See proof of Theorem 1)} \tag{1.3}$$

Whether or not there are infinitely many twin prime pairs in the Gregory-Leibniz series remains unknown.

In this note, the value of the generalized infinite series

$$S_k = \sum_{n=1}^{\infty} 1/((k \cdot n)^2 - 1), \ k = 3, 4, 5, \ \ldots \tag{1.4}$$

is obtained using relationships among the zeta function $\zeta(2m)$, the Bernoulli coefficients, $B(2m)$, and the Laurent expansion of $\cot(x)$ [AS, BS]. A proof of the Gregory-Leibniz

series (1.1) is given, and series for π that involve $\sqrt{2}$, $\sqrt{3}$, and the sums S_6, S_{12}, S_{24}, and S_{48} are found.

2. Theorems

Theorem 2.1

The infinite sum

$$S_k = \sum_{n=1}^{\infty} 1/((k \bullet n)^2 - 1), \ k = 3, 4, 5, \ldots$$

$$= [1 - (\pi/k)cot(\pi/k)]/2$$

Proof.

Let

$$S_k = \sum_{n=1}^{\infty} 1/((k \bullet n)^2 - 1), \ k = 3, 4, 5, \ldots \tag{2.1}$$

Factoring out $1/(k \bullet n)^2$, and expanding $(1 - (1/(k \bullet n)^2))^{-1}$ gives

$$S_k = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 1/(k \bullet n)^{2m} \tag{2.2}$$

The zeta function $\zeta(2m)$ relates the sum of terms $1/N^{2m}$ to the Bernoulli coefficients, B_{2m} [AS]

$$\zeta(2m) = \sum_{N=1}^{\infty} 1/N^{2m} = (2\pi)^{2m} |B_{2m}| /(2 \bullet (2m)!), \ m = 1, 2, 3 \ldots \tag{2.3}$$

We can therefore write

$$S_k = \sum_{m=1}^{\infty} 2^{2m} |B_{2m}| (\pi/k)^{2m} /(2 \bullet (2m)!) \tag{2.4}$$

The Laurent series for $cot(x)$, written in terms of the Bernoulli coefficients B_{2m}, is [AS]

$$cot(x) = (1/x) - \sum_{m=1}^{\infty} (2^{2m} |B_{2m}| x^{2m})/(x \bullet (2m)!) \ \text{for} \ 0 < x < \pi \tag{2.5}$$

Putting $x = \pi/k$, we therefore obtain

$$cot(\pi/k) - (k/\pi) = - \sum_{m=1}^{\infty} (2^{2m} |B_{2m}| (\pi/k)^{2m} (k/\pi))/(2m)! \tag{2.6}$$

giving

$$S_k = [1 - (\pi/k)cot(\pi/k)]/2, \ k = 3, 4, 5, \ldots \qquad \square \tag{2.7}$$

Some values of S_k are

$$S_4 = 0.107300918301275845 \ldots$$
$$S_6 = 0.046550158941445537 \ldots$$
$$S_{12} = 0.011475691671573332 \ldots$$
$$S_{24} = 0.002859055853921023 \ldots$$
$$S_{48} = 0.000714151049012813 \ldots$$

The product of every twin prime pair $p(p + 2)$ contributes to the value of the infinite series

$$S_6 = 0.046550158941445537 \ldots$$

(The result

$$\pi \cot(\pi x) = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2}$$

is given in Weisstein [Wei] with neither proof nor reference).

Theorem 2.2

$$1 - 2S_4 = \sum_{n=0}^{\infty} (-1)^n /(2n + 1), \text{ the Gregory-Leibniz series.}$$

Proof.

Using (2.7), we have

$$2S_4 = 1 - (\pi/4)\cot(\pi/4)$$
$$= 1 - (\pi/4) \tag{2.8}$$

therefore

$$\pi/4 = 1 - 2 \sum_{n=1}^{\infty} 1/((4n)^2 - 1) \tag{2.9}$$

$$= 1 - 2\{(1/15) + (1/63) + (1/143) + (1/255) + \ldots\}$$
$$= 1 - \{[(1/3) - (1/5)] + [(1/7) - (1/9)] + \ldots\}$$
$$= 1 - (1/3) + (1/5) - (1/7) + (1/9) - \ldots$$
$$= \sum_{n=0}^{\infty} (-1)^n /(2n + 1). \qquad \square \tag{2.10}$$

This proof is fundamentally different from the traditional approach that involves expanding the function $1/(1 + x^2)$ as a power series, integrating term-by-term from 0 to 1

(including the remainder after n-terms), letting n → ∞, and equating the result to arctan(x). In such an approach, it is necessary to prove that the method is valid when x = 1 in order to use arctan(1) = π/4.

3. Series for π

From (2.7), we obtain

$$\pi = k \cdot \tan(\pi/k)[1 - 2S_k], \quad k = 3, 4, 5, \ldots \tag{3.1}$$

If the number field is extended to include $\sqrt{2}$ and $\sqrt{3}$, the half-angle formulae

$$\sin(\theta/2) = \pm[(1 - \cos(\theta))/2]^{1/2} \text{ and } \cos(\theta/2) = \pm[(1 + \cos(\theta))/2]^{1/2} \tag{3.2}$$

can be used to give successive values of $\tan(\pi/(6 \cdot 2^j))$, j = 1, 2, 3, …

where

$$\tan(\pi/6) = 1/\sqrt{3} \tag{3.3}$$

For example,

$$\tan(\pi/12) = 1/(2 + \sqrt{3})$$

$$\tan(\pi/24) = (a - b)/(b - 2), \quad a = 2\sqrt{2} \text{ and } b = 1 + \sqrt{3}$$

and

$$\tan(\pi/48) = (2a/(a - b))^{1/2} - (b - 2)/(a - b). \tag{3.4}$$

Using the expression for $\tan(\pi/48)$, we obtain

$$\pi = 48 \times 0.0655435 \ldots (1 - 2S_{48})$$

$$= 3.1460862 \ldots (1 - 2 \times 0.0007142 \ldots)$$

$$= 3.1415926 \ldots$$

References

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[BS]   I. N. Bronshtein and K. A. Semendyayev, *Handbook of Mathematics*, Van Nostrand Reinhold Co. New York (1985)

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[Wei] Eric W. Weisstein, "Cotangent" from *Mathworld*, a Wolfram Web Resource, http://mathworld.wolfram.com/cotangent.html